Journal of Statistical Physics, Vol. 106, Nos. 3/4, February 2002 (© 2002)

Measure Zero Spectrum of a Class of Schrödinger Operators

Qing-Hui Liu,[1] Bo Tan,[2] Zhi-Xiong Wen,[2] and Jun Wu[2]

Received June 28, 2001; accepted September 24, 2001

We study the measure of the spectrum of a class of one-dimensional discrete Schrödinger operators $H_{v,\omega}$ with potential $v(\omega)$ generated by any primitive substitutions. It is well known that the spectrum of $H_{v,\omega}$ is singular continuous.[(1)] We will give a more exact result that the spectrum of $H_{v,\omega}$ is a set of Lebesgue measure zero, by removing two hypotheses (the semi-primitive of a certain induced substitution and the existence of square word) from a theorem due to Bovier and Ghez.[(2)]

KEY WORDS: Schrödinger operator; spectrum; primitive substitution; trace map.

1. INTRODUCTION

Since the discovery of quasi-crystal,[(3)] the spectral properties of one-dimensional discrete Schrödinger operator with quasi-periodic potential have attracted considerable attention during the past two decade. We study the Schrödinger operator with potential generated by primitive substitution.

Firstly we define sequence generated by primitive substitution (see for example, refs. 1, 2, 4, and 5 for details).

Let $\mathscr{A}$ be an alphabet, ξ be an primitive substitution over $\mathscr{A}$. Choose a fixed point z^+ of ξ. Taking any $z \in \mathscr{A}^{\mathbb{Z}}$ satisfying $z_n = z_n^+$ for $n \geqslant 0$ and defining Ω_ξ as the set of accumulation points of $\{T^n z : n \in \mathbb{N}\}$, where T is the left shift on $\mathscr{A}^{\mathbb{Z}}$. As claimed in refs. 1, 5, and 6, (Ω_ξ, T) is uniquely ergodic (i.e., there exists only one T-invariant measure) and minimal (i.e.,

[1] Department of Mathematics, Nanjing University, Nanjing, Jiangsu, People's Republic of China; e-mail: gzwqhl@263.net

[2] Laboratory of Pure and Applied Mathematics, WuhanUniversity, Wuhan 430072, Hubei, People's Republic of China; e-mail: tanbo@colmath.whu.edu.cn, zhxwen@whu.edu.cn, wujunyu@public.wh.hb.cn

the orbit of every $\omega \in \Omega_\xi$ is dense in Ω_ξ), so it is strictly ergodic. Ω_ξ is independent of the selection of fixed point. Suppose that Ω_ξ is not finite, then any $\omega \in \Omega_\xi$ is not periodic.

We call any $\omega \in \Omega_\xi$ a *sequence generated by primitive substitution* ξ. By minimality of (Ω_ξ, T), the spectrum of $H_{v,\omega}$ are the same for all $\omega \in \Omega_\xi$, and we denote the set by Σ_ξ.

Fix an alphabet $\mathscr{A}$ and a primitive substitution ξ over $\mathscr{A}$. Let v be an injective map from the finite set $\mathscr{A}$ to $\mathbb{R}$, $\omega \in \Omega_\xi$, we define $H_{v,\omega}$ as

$$(H_{v,\omega}\phi)_n = -\phi_{n+1} - \phi_{n-1} + v(\omega_n)\,\phi_n, \qquad n \in \mathbb{Z}, \quad \phi \in l^2(\mathbb{Z}), \tag{1}$$

which is called a Schrödinger operator with potential generated by primitive substitution ξ.

It was proved by Hof, Knill and Simon,[1] that, for any primitive substitution ξ, and any $\omega \in \Omega_\xi$, Σ_ξ is singular continuous (note that a singular spectrum may have positive Lebesgue measure. They applied a result proved by Hof[5] that, for any $E \in \mathbb{C}$, the Lyapunov exponent $\gamma_{v,\omega}(E)$ exists and are equal for any $\omega \in \Omega_\xi$. It is known that, $\gamma_{v,\omega}(E) > 0$ for $E \in \mathbb{C} \setminus \Sigma_\xi$, and $\gamma_{v,\omega}(E) \geqslant 0$ for any $E \in \Sigma_\xi$. However, if $\gamma_{v,\omega}(E) = 0$ for any $E \in \Sigma_\xi$, then by a theorem of Kotani,[7] Σ_ξ is a set of Lebesgue measure zero.

Let us recall two papers that had ever estimated the related Lyapunov exponent. It was proved by Bellissard *et al.*[8] that for ω generated by a class of circle map, which is the usually called Sturmian sequence (some Sturmian sequences can be generated by invertible substitution over two-letter alphabet), $\gamma_{v,\omega}(E) = 0$ for any $E \in \sigma(H_{v,\omega})$. For non-periodic ω generated by more general primitive substitution, Bovier and Ghez[2] provided a way to prove zero Lyapunov exponent for E in the spectrum, if the substitution satisfies two hypotheses they gave. What we want to do in this paper is to remove the two hypotheses, so for any non-periodic ω generated by primitive substitution, $\sigma(H_{v,\omega})$ is a set of Lebesgue measure zero. Moreover, it was proved by Sütö[9] such that Σ_ξ contains no isolated point, so it is also a Cantor set.

We note that absence of isolated point does not mean that Σ_ξ contains no point spectrum of $H_{v,\omega}$ for any $\omega \in \Omega_\xi$. Uniform absence of point spectrum for Sturmian potentials is closely studied by Damanik and Lenz recently (see ref. 10 and reference therein).

Our work is a continuation of Casdagli,[11], Sütö,[12] Bellissard *et al.*,[8] Bovier and Ghez,[2] in the sense of applying the so called trace polynomial.

Casdagli[11] proved that pseudo-spectrum B_∞ of Schrödinger operator with potential to be Fibonacci sequence is a Cantor set. B_∞ constructs a bridge to apply the trace map theory.

Sütö[12] proved that $\sigma(H_{v,\omega})$ is a Cantor set for some v and ω the Fibonacci sequence. He established a relation between B_∞ and $\sigma(H_{v,\omega})$ by periodical approach, which can also be extended to any primitive substitution by choosing a suitable way of generation.

Bellissard *et al.*[8] proved that $\sigma(H_{v,\omega})$ is a Cantor set of zero Lebesgue measure, for any $\omega \in \Omega_\tau$ with τ an arbitrary invertible substitution over two letter alphabet. They applied a result of Kotani,[7] and discussed all sequence in Ω_τ instead of a single sequence.

Bovier and Ghez[2] gave a general result under two hypotheses:

Theorem A1. Let ξ be a primitive substitution with non-periodic substitution sequence on a finite alphabet $\mathscr{A}$. Let v be a non-constant map from $\mathscr{A}$ to $\mathbb{R}$, ω a two-side infinite sequence generated by ξ, and $H_{v,\omega}$ be the Schrödinger operator defined in (1). Suppose there exists a trace map whose induced substitution ϕ, defined on an alphabet $\mathscr{B}$, is semi-primitive. Assume further that there exists $k < \infty$ such that $\xi^k(a)$ contains the word $\beta\beta$ for some $\beta \in \mathscr{B}$. Then the spectrum of $H_{v,\omega}$ is singular and supported on a set of zero Lebesgue measure.

Where they applied leading term of trace polynomial to estimate trace, introduced the semi-primitivity of induced substitution, and applied some methods of Casdagli, Sütö, and Bellissard *et al.*

Note that in ref. 2, the generation of a two-side infinite sequence from a primitive substitution, say ξ, is different from the generation we give above, which caused some trouble and is modified in ref. 13. And it may still happen that the generated sequence ω' they given in ref. 13 is not in the set Ω_ξ. In fact, it can be proved that the spectrum for this kind of potential is still Lebesgue measure zero, but we can not expect that it is equal to Σ_ξ.

They checked the hypotheses of semi-primitivity and existence of square word for some substitutions, and they found that all the examples they had checked satisfy the two hypotheses, except that the corresponding induced substitution of Rudin–Shapiro substitution is not semi-primitive. But we found that they had made some errors on it. The trace map they chose is from ref. 14, while some trace polynomial in the trace map has no leading term for $\mathscr{A}$-degree defined themselves, this may cause trouble in their estimating. Moreover, we would like to note that since trace map of a substitution over three or more letter alphabet is not unique, absence of semi-primitivity for the induced substitution of one trace map associated with a primitive substitution does not imply that of any other trace map.

To assure the existence of leading term, we will focus our attention on a class of special trace polynomials, which is called natural trace polynomial in this paper. Even the natural trace polynomial may be not unique for word over three or more letter alphabet. So it may need complex and tedious calculation to justify whether there exists semi-primitive induced substitution for a given primitive substitution. In fact we had made out a trace map associated with Rudin–Shapiro substitution, of which the induced substitution is semi-primitive. But this is not yet important, since for any primitive substitution ξ over any finite alphabet, we will prove in Section 3 that there exist $l > 0$ and suitable trace map associated with ξ^l (note that $\Omega_{\xi^l} = \Omega_\xi$, since their substitution sequences are equal) such that the corresponding induced substitutions are semi-primitive.

In ref. 2, the existence of square word is used to estimate the Lyapunov exponent. But it is not a trivial condition, since for substitution sequences on three or more letter alphabet, the hypothesis may fail. For example the substitution defined by $\xi(a) = abc$, $\xi(b) = ac$, $\xi(c) = b$ is a counter-example.[15] We will remove the hypothesis of existence of square word through further analysis.

In this paper, we will not prove the theorems which we needed and had already been proved in ref. 2. So combining with these theorems, we prove in Section 4 that

Theorem 1.1. Let ξ be a primitive substitution with non-periodic substitution sequence on a finite alphabet $\mathscr{A}$. Let v be an injective map from $\mathscr{A}$ to $\mathbb{R}$, Ω a set of all two-side infinite sequence generated by ξ, for any $\omega \in \Omega$, let $H_{v,\omega}$ be the Schrödinger operator defined in (1), and let $\Sigma_\xi = \sigma(H_{v,\omega})$ be the spectrum of $H_{v,\omega}$. Then Σ_ξ is a Cantor set of zero Lebesgue measure.

Note. We need the substitution sequence to be non-periodic since there are primitive substitutions, say for example $\xi(a) = a(ba)^n$, $\xi(b) = b(ab)^m$, with periodic substitution sequence and the spectrum for such potential is purely absolutely continuous.

2. NATURAL TRACE POLYNOMIAL

Fix an alphabet $\mathscr{A} = \{\alpha_1, \alpha_2,\ldots, \alpha_m\}$. Let $K = \sum_{i=1}^{m} \frac{m!}{i\cdot(m-i)!}$. Choose a set $\mathscr{B} = \{\beta_1, \beta_2,\ldots, \beta_K\} \subset \mathscr{A}^*$ satisfying that, every word in $\mathscr{B}$ contains no letter in $\mathscr{A}$ twice and no word in $\mathscr{B}$ is a cyclic permutation of the other.

Let $SL(2, \mathbb{R})$ be the set of all real 2×2 matrices with determinant 1, $\mathrm{Hom}(\mathscr{A}^*, SL(2, \mathbb{R}))$ be the set of all the homomorphism maps from $\mathscr{A}^*$ into $SL(2, \mathbb{R})$. Any $\varphi \in \mathrm{Hom}(\mathscr{A}^*, SL(2, \mathbb{R}))$ is determined by matrices $\{\varphi(\alpha_i)\}_{i=1}^m$, and for $u = u_1 u_2 \cdots u_l (u_i \in \mathscr{A})$,

$$\varphi(u) = \varphi(u_1)\, \varphi(u_2) \cdots \varphi(u_l),$$

thus $\varphi(u)$ can be seen as a matrix got by changing every letter in u to the associated matrix. We call $\varphi \in \mathrm{Hom}(\mathscr{A}^*, \mathrm{SL}(2, \mathbb{R}))$ a representation.

Let $x_{\beta_1},\dots, x_{\beta_K}$ be K variables. Let $u \in \mathscr{A}^*$. A polynomial $P \in \mathbb{R}[x_{\beta_1},\dots, x_{\beta_K}]$ is said to be a *trace polynomial* of u, if for any representation φ, we have

$$\operatorname{tr} \varphi(u) = P(\operatorname{tr} \varphi(\beta_1), \operatorname{tr} \varphi(\beta_2),\dots, \operatorname{tr} \varphi(\beta_K)),$$

where $\operatorname{tr}(M)$ denotes the trace of the matrix M. For example, x_{β_i} is a trace polynomial of β_i.

Bovier and Ghez[2] introduced $\mathscr{A}$-degree, by define that $\mathrm{d}_{\mathscr{A}}\, x_{\beta_i} = |\beta_i|$, and for any $P, Q \in \mathbb{R}[x_{\beta_1},\dots, x_{\beta_K}]$,

$$\mathrm{d}_{\mathscr{A}}(P+Q) = \max\{\mathrm{d}_{\mathscr{A}}\, P, \mathrm{d}_{\mathscr{A}}\, Q\}, \qquad \mathrm{d}_{\mathscr{A}}\, PQ = \mathrm{d}_{\mathscr{A}}\, P + \mathrm{d}_{\mathscr{A}}\, Q.$$

This kind of degree construct a connection between the length of a word and the following kind of trace polynomial.

In general we construct trace polynomial by repeatedly using the following two equation, that is for any matrices $A, B \in \mathrm{SL}(2, \mathbb{R})$,

$$\operatorname{tr} AB = \operatorname{tr} BA, \qquad ABA = (\operatorname{tr} AB)\, A + B - (\operatorname{tr} B)\, I \tag{2}$$

where I is unit 2×2 matrix. We refer to refs. 2 and 4 for detail of the construction. For any word $u \in \mathscr{A}^*$, we call a trace polynomial of u which can be got in this way a *natural trace polynomial* of u. It's readily to obtain the following properties of natural trace polynomials.

Property 2.1. Let P be a natural trace polynomial of $u \in \mathscr{A}^*$, and

$$P = \sum_{i=1}^{l} c_i P_i, \qquad c_i \neq 0$$

$$\mathrm{d}_{\mathscr{A}}\, P_i \geqslant \mathrm{d}_{\mathscr{A}}\, P_{i+1}, \qquad i = 1, 2,\dots, l-1,$$

where P_i is monomial with coefficient 1. We have

(i) $c_1 = 1$,

(ii) $L[W(P_1)] > L[W(P_i)]$, $i = 2, 3,\dots, l$,

(iii) $L[W(P_1)] = L[u]$,

(iv) $\mathrm{d}_{\mathscr{A}}\, P = |u|$,

where for any $v \in \mathscr{A}^*$, $L(v)$ is the language[4] of v defined by

$$L(v) = (|v|_{\alpha_1}, |v|_{\alpha_2},\dots, |v|_{\alpha_m}), \qquad \alpha_i \in \mathscr{A},$$

which is an $|\mathscr{A}|$-dimensional vector, $|v|_{\alpha_i}$ is the number of letter α_i contained in v. The ">" between the vectors defined as follows, if $x=(x_1,\dots,x_m)$ and $y=(y_1,\dots,y_m)$, then $x>y$ means that $x_i \geqslant y_i$ and there exists $1\leqslant j\leqslant m$ such that $x_j>y_j$.

Note 1. By Property 2.1(ii), we know that the $\mathscr{A}$-degree of P_1 is greater than any other P_i. We call P_1 the leading term of P, denoted as $\tilde{P}$. Moreover, without Property 2.1(ii), some estimating on trace map in ref. 2 may be fail if we use only the fact that $\mathrm{d}_{\mathscr{A}}\, P_1$ is greater than any other $\mathrm{d}_{\mathscr{A}}\, P_i$.

Note 2. For any word u over three or more letter alphabet, even the natural trace polynomial may be not unique, and their leading term may be different. For example, let $u=abcacb$, by the method of constructing natural trace polynomial, the partitions $a(bc)\,a(cb)$ and $c(a)\,c(bab)$ result different natural trace polynomials

$$P=x_{abc}x_{acb}+x_bx_cx_{bc}-x_b^2-x_c^2-x_{bc}^2+2,$$

$$Q=x_{ac}x_{bc}x_{ab}+x_a^2+x_{ac}^2+x_{ab}^2-x_ax_cx_{ac}-x_ax_bx_{ab}-2.$$

So $\tilde{P}=x_{abc}x_{acb}$ and $\tilde{Q}=x_{ab}x_{ac}x_{bc}$.

Note 3. Not any trace polynomial has above properties. For any word over two letter alphabet, its trace polynomial is unique and is also a natural trace polynomial. But for any word over three or more letter alphabet, its trace polynomial is not unique and may not be natural trace polynomial. In fact, for three or more letter alphabet $\mathscr{A}$, there exist polynomial Λ over $\mathbb{R}^K$ such that for any representation φ,

$$\Lambda(\operatorname{tr}\varphi(\beta_1), \operatorname{tr}\varphi(\beta_2),\dots, \operatorname{tr}\varphi(\beta_K))\equiv 0,$$

see ref. 4 for $|\mathscr{A}|=3$ and ref. 16 for $|\mathscr{A}|>3$. This implies that if P is a trace polynomial of a word u over $\mathscr{A}$ then $P+\Lambda^n$ is also a trace polynomial of u for any non-negative integer n. For sufficiently large n, the $\mathscr{A}$-degree of $P+\Lambda^n$ is not $|u|$, and $P+\Lambda^n$ has no leading term.

3. SEMI-PRIMITIVE OF INDUCED SUBSTITUTION

Let ξ be a primitive substitution over $\mathscr{A}$. Associate a polynomial map with ξ as follows,

$$\Phi(x)=(\Phi_{\beta_1}(x), \Phi_{\beta_2}(x),\dots, \Phi_{\beta_K}(x)), \qquad x\in\mathbb{R}^K$$

where we choose Φ_{β_i} to be a natural trace polynomial of $\xi(\beta_i)$. The polynomial map Φ is a usually called trace map associated with ξ. Such trace map may be not unique, depending on the natural trace polynomials we choose.

Now we introduce the reduced trace map and induced substitution and semi-primitivity defined in ref. 2.

Since any Φ_{β_i} has leading term $\tilde{\Phi}_{\beta_i}$ of coefficient 1, we can define the reduced map of Φ on $\mathbb{R}^K$ by,

$$\tilde{\Phi}(x) = (\tilde{\Phi}_{\beta_1}(x), \tilde{\Phi}_{\beta_2}(x),..., \tilde{\Phi}_{\beta_K}(x)), \qquad x \in \mathbb{R}^K,$$

and a induced substitution ϕ of $\tilde{\Phi}$, $\phi : \mathscr{B} \mapsto \mathscr{B}^*$:

$$\phi(\beta) = W(\tilde{\Phi}_\beta), \qquad \forall \beta \in \mathscr{B},$$

where $W(\tilde{\Phi}_\beta)$ is the word over $\mathscr{B}$ associated with the monomial $\tilde{\Phi}_\beta$, defined by for any monomial on $\mathbb{R}^K$ of the form $P = x_{\beta_{i_1}} x_{\beta_{i_2}} \cdots x_{\beta_{i_l}}$, where $\beta_{i_j} \in \mathscr{B}$, $j = 1, 2,..., l$,

$$W(P) = \beta_{i_1} \beta_{i_2} \cdots \beta_{i_l} \in \mathscr{B}^*. \tag{3}$$

$\phi(\beta) = W(\tilde{\Phi}_\beta)$ can only be determined up to the order of the letters in $\phi(\beta)$. But what we consider is the semi-primitivity of ϕ, which is independent of the order of letters, and if one induced substitution of $\tilde{\Phi}$ is semi-primitive, so is the other.

Definition. A substitution ϕ over $\mathscr{B}$ is said to be semi-primitive if

(i) there exists a subset $\mathscr{C} \subset \mathscr{B}$, such that $\forall \beta \in \mathscr{C}$, $\phi(\beta)$ contains only letters in $\mathscr{C}$, and $\phi|_{\mathscr{C}}$ is primitive (i.e., there exists $k \in \mathbb{N}$ such that for any $\beta \in \mathscr{C}$, $\phi^k(\beta)$ contains all letters in $\mathscr{C}$);

(ii) there exists $l \in \mathbb{N}$, such that for all $\beta \in \mathscr{B}$, $\phi^l(\beta)$ contain letters in $\mathscr{C}$.

We give a sufficient and necessary condition for a substitution to be semi-primitive.

Lemma 3.1. A substitution ϕ over an alphabet $\mathscr{B}$ is semi-primitive if and only if there exists a letter $\beta_0 \in \mathscr{B}$ and $k > 0$ such that for any letter $\beta \in \mathscr{B}$, $\phi^k(\beta)$ contains β_0 and the length of $\phi^n(\beta_0)$ tends to infinity with n.

Proof. It is readily to prove the necessary part, so we prove the sufficient part in the following. Without loose of generality, suppose $k = 1$, otherwise we can consider ϕ^k.

For any integer n and $\beta \in \mathscr{B}$, let $N(\beta, n)$ be the number of β contained in the word $\phi^n(\beta_0)$, i.e.,

$$N(\beta, n) := |\phi^n(\beta_0)|_\beta.$$

The number $N(\beta, n)$ is increasing with n. Since β_0 is contained in $\phi(\beta_0)$, $\phi^n(\beta_0)$ is a subword of $\phi^{n+1}(\beta_0)$. Let

$$\mathscr{C} := \{\beta \in \mathscr{B} : \lim_{n\to\infty} N(\beta, n) = \infty\}.$$

Then we have

(a) $\mathscr{C}$ is not empty, since the length of $\phi^n(\beta_0)$ tends to infinity with n;

(b) For any $\gamma \in \mathscr{C}$, $\phi(\gamma)$ contains only letters in $\mathscr{C}$: suppose that a letter $\beta \in \mathscr{B}$ is contained in $\phi(\gamma)$, then $N(\beta, n+1) \geqslant N(\gamma, n)$, so $\beta \in \mathscr{C}$;

(c) There exists $l > 0$ such that for any $\beta \in \mathscr{B}$, $\phi^{l+1}(\beta)$ contains all letters in $\mathscr{C}$. The reason is that there exists $l > 0$ such that $\phi^l(\beta_0)$ contains all letters in $\mathscr{C}$, and $\phi^l(\beta_0)$ is a subword of $\phi^{l+1}(\beta)$.

(a), (b), and (c) implies that ϕ is semi-primitive. ▌

It is a direct corollary of the lemma that the induced substitution corresponding to a primitive substitution over two-letter alphabet (say $\mathscr{A} = \{a, b\}$) is semi-primitive, since we only need to put $\beta_0 = ab$ (in this case $\mathscr{B} = \{a, b, ab\}$). But for three or more letter alphabet, it is not so obvious. We prove the following theorem by choosing suitable trace map associated with primitive substitution.

Theorem 3.2. If ξ is primitive, then there exists $l \in \mathbb{N}$ and a trace map associated with ξ^l, such that the corresponding induced substitution is semi-primitive.

Proof. Since ξ is primitive, there exists an integer t such that for any $\alpha \in \mathscr{A}$, $\xi^t(\alpha)$ contains at least two a's.

We choose a suitable trace map associated with ξ^{2t}.

(i) Since $\xi^t(a)$ contains at least two a, we can write $\xi^t(a) = aw_1aw_2$, where $w_1, w_2 \in \mathscr{A}^*$. We fix a natural trace polynomial P_{aw_1} of aw_1, and fix a letter $\beta_0 \in \mathscr{B}$ which is contained in the word $W(\tilde{P}_{aw_1}) \in \mathscr{B}^*$.

(ii) For any $\beta \in \mathscr{B}$, since $\xi^t(\beta)$ contains at least one a, $\xi^{2t}(\beta)$ contains aw_1a, then we can choose a cyclic permutation of $\xi^{2t}(\beta)$ of the form aw_1aw_β. Since for any representation φ,

$$\begin{aligned}\operatorname{tr}\varphi[\xi^{2t}(\beta)] &= \operatorname{tr}\varphi(aw_1aw_\beta)\\ &= \operatorname{tr}\varphi(aw_1)\operatorname{tr}\varphi(aw_\beta) + \operatorname{tr}\varphi(w_1w_\beta) - \operatorname{tr}\varphi(w_1)\operatorname{tr}\varphi(w_\beta)\end{aligned}$$

we choose natural trace polynomials P_{aw_β}, $P_{w_1 w_\beta}$, P_{w_1}, P_{w_β} of the words aw_β, $w_1 w_\beta$, w_1, w_β respectively, then

$$\Phi_\beta := P_{aw_1} P_{aw_\beta} + P_{w_1 w_\beta} - P_{w_1} P_{w_\beta}$$

is a natural trace polynomial of $\xi^{2t}(\beta)$. Thus we get a trace map Φ associated with the substitution ξ^{2t}, and the reduced map $\tilde{\Phi}$ of Φ with

$$\tilde{\Phi}_\beta = \tilde{P}_{aw_1} \tilde{P}_{aw_\beta}.$$

Set an induced substitution of $\tilde{\Phi}$ by

$$\phi(\beta) = W(\tilde{P}_{aw_1})\, W(\tilde{P}_{aw_\beta}) = W_{aw_1} W(\tilde{P}_{aw_\beta}). \tag{4}$$

For any $\beta \in \mathscr{B}$, $W(\tilde{P}_{aw_1})$ is a subword of $\phi(\beta)$, and so β_0 is contained in $\phi(\beta)$.

Since $L[\phi^n(\beta_0)] = L[\xi^{2tn}(\beta_0)]$, the length of $\phi^n(\beta_0)$ as a word in $\mathscr{B}^*$ also tends to infinity with n. So according to Lemma 3.1, ϕ is semi-primitive. ∎

4. PROOF OF THEOREM 1.1

Now to prove Theorem 1.1, by the results proved by Bovier and Ghez,[2] we only need to get rid of the condition of having a square word from Theorem A1.

The condition of having a square word is only used by Bovier and Ghez[2] to show that whenever the traces diverge more slowly than exponential, then so does the norm of the transfer operator. In fact, if there exists $k > 0$, such that $\xi^k(a)$ contains square word, i.e., $\xi^k(a) = uw^2v$, then

$$\|\varphi(\xi^n(a))\| \leqslant |\mathrm{tr}\, \varphi(\xi^{n-k}(w))|\, \|\xi^{n-k}(uwv)\| + \|\xi^{n-k}(uv)\|. \tag{5}$$

It is this equation that implies that whenever the traces diverge more slowly than exponential, then so does the norm of the transfer operator.

As stated in Section 1, the existence of square word in the substitution sequence could fail, thus we can not use Eq. (5) to get the desired result directly. But we have the following facts. By the primitivity of ξ, there exists $k > 0$ such that $\xi^k(a)$ contains at least two a's, then $\xi^k(a)$ can be written as $uawav$, by Eq. (2) we have that

$$\|\varphi(\xi^n(a))\| \leqslant |\mathrm{tr}\, \varphi(\xi^{n-k}(aw))|\, \|\xi^{n-k}(uav)\| + \|\xi^{n-k}(uwv)\| + |\mathrm{tr}\, \varphi(\xi^{n-k}(w))|\, \|\xi^{n-k}(uv)\|,$$

this equation also implies that whenever the traces diverge more slowly than exponential, then so does the norm of the transfer operator. So Theorem A1 is also valid if we get rid of the condition of having square word.

Let ξ be a primitive substitution with non-periodic substitution sequence on a finite alphabet $\mathscr{A}$. By Theorem 3.2, there exists an integer $l > 0$ and a trace polynomial of ξ^l such that the induced substitution is semiprimitive. Then along the line of proof of Theorem A1 in ref. 2, getting rid of the condition of having square word we prove Theorem 1.1.

ACKNOWLEDGMENTS

This work was supported by the Special Funds for Major State Basic Research. The authors thank Professor Zhi-Ying Wen for helpful discussions and advices and thank the referee for his suggestions.

REFERENCES

1. A. Hof, O. Knill, B. Simon, Singular continuous spectrum for palindromic Schrödinger operators, *Commun. Math. Phys.* **174**:149–159 (1995).
2. A. Bovier and J.-M. Ghez, Spectrum properties of one-dimensional Schrödinger operators with potentials generated by substitutions, *Commun. Math. Phys.* **158**:45–66 (1993).
3. D. Schechtman, I. Blech, D. Gratias, and J. Cahn, Metallic phase with long-range orientational order and no translational symmetry, *Phys. Rev. Lett.* **53**:1951–1954 (1984).
4. J. Peyrière, Z.-X. Wen, et Z.-Y. Wen, Polynomês associées aux endomorphismes de groupes libres, *L'Enseig. Math.* **t. 39**:153–175 (1993).
5. A. Hof, Some remarks on discrete aperiodic Schrödinger operator, *J. Statist. Phys.* **72**:1353–1374 (1993).
6. M. Queffélec, *Substitution Dynamical System—Spectral Analysis*, Lecture Notes in Mathematics, Vol. 1294 (New York, Springer-Verlag, 1987).
7. S. Kotani, Jacobi matrices with random potentials taking finitely many values, *Rev. Math. Phys.* **1**:129–133 (1990).
8. J. Bellissard, B. Iochum, E. Scoppola, and D. Testart, Spectral properties of one dimensional quasi-crystals, *Commun. Math. Phys.* **125**:527–543 (1989).
9. A. Sütö, Schrödinger difference equation with deterministic ergodic potentials, *Beyond Quasicrystals*, F. Alex and D. Gratias, eds. (Springer-Verlag, 1994).
10. D. Damanik and D. Lenz, Uniform spectral properties of one-dimensional quasicrystals, I. Absence of eigenvalues, *Commun. Math. Phys.* **207**:687–696 (1999).
11. M. Casdagli, Symbolic dynamics for the renormalization map of quasipeoriodic Schrödinger equation, *Commun. Math. Phys.* **107**:295–318 (1986).
12. A. Süto, The spectrum of a quasipeoriodic Schödinger operator, *Commun. Math. Phys.* **111**:409–415 (1987).
13. A. Bovier and J.-M. Ghez, Erratum Spectrum properties of one-dimensional Schrödinger operators with potentials generated by substitutions, *Commun. Math. Phys.* **166**:431–432 (1994).

14. M. Kolář and F. Nori, Trace maps of general substitutional sequences, *Phys. Rev.* **B42**:1062–1065(1990).
15. M. Lothaire, *Combinatorics on Words*, Encyclopedia of Mathematics and Its Applications, Vol. 17.
16. Z.-X. Wen, Relations polynomiales entre les traces de produits de matrices, Série I, *C. R. Acad. Sci. Paris*, **t. 318**:99–104 (1994).
17. Z.-X. Wen and Z.-Y. Wen, On the leading term and the degree of the polynomial trace mapping associated with a substitution, *J. Statist. Phys.* **75**:627–641 (1994).